

NON-NORMAL NUMBERS WITH RESPECT TO INFINITE MARKOV PARTITIONS

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ABSTRACT. In the present paper we investigate two special types of non-normal numbers. On the one hand we call a number extremely non-normal if the set of accumulation points of its frequency of blocks vector is the full set of shift invariant probability vectors. On the other hand we call a number having maximal oscillation frequency if for any fixed block the set of accumulation points of its frequency vector is the full set of possible probability vectors. The goal is to investigate the Baire category of these numbers for infinite Markov partitions.

1. INTRODUCTION

Let \mathbb{I} denote the irrational numbers in the unit interval, *i.e.*, $\mathbb{I} := [0, 1] \setminus \mathbb{Q}$. Then every $x \in \mathbb{I}$ can be represented in a unique way as an infinite simple continued fraction, namely

$$x = [a_1(x), a_2(x), a_3(x), \dots] = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}},$$

where $a_k(x) \in \mathbb{N}$ for $k \geq 1$.

In the present paper we investigate the limiting frequency of blocks of $a_k(x)$. For $k \geq 1$ a positive integer and a block $\mathbf{b} = b_1 \dots b_k \in \mathbb{N}^k$ we denote by $\Pi(x, \mathbf{b}, n)$ the number of occurrences of this block \mathbf{b} among the first n digits of x , *i.e.*

$$\Pi(x, \mathbf{b}, n) := \frac{\#\{0 \leq i < n : a_{i+1}(x) = b_1, \dots, a_{i+k}(x) = b_k\}}{n}.$$

Furthermore we denote by

$$\Pi_k(x, n) = (\Pi(x, \mathbf{b}, n))_{\mathbf{b} \in \mathbb{N}^k}$$

the vector of frequencies $\Pi(x, \mathbf{b}, n)$ for all blocks \mathbf{b} of length k .

For digits (blocks of length 1) a famous result of Lévy [12] states that for Lebesgue almost all $x \in \mathbb{I}$ we have

$$(1.1) \quad \Pi(x, b, n) \rightarrow \frac{1}{\log 2} \log \frac{(b+2)^2}{b(b+2)}$$

for all $b \in \mathbb{N}$. In analogy with normal numbers in q -adic number systems (*cf.* [11] or [8]) we call a $x \in \mathbb{I}$ simple (continued-fraction-)normal if it satisfies (1.1).

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We can extend this notion to (continued-fraction-)normal numbers by using the Gauss measure defined by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx,$$

where $A \subset [0, 1]$ is a Lebesgue measurable set. Then we call a number (continued-fraction-)normal if the asymptotic frequency of its block of digits is determined by the Gauss measure. Now an application of Birkhoff's ergodic theory (*cf.* [5] or [6]) yields that almost all numbers are (continued-fraction-)normal with respect to the Lebesgue measure.

In the present paper we want to focus on numbers which are described by their frequency of blocks of digits. For each $k \geq 1$, we define the simplex of all probability vectors Δ_k by

$$\Delta_k = \left\{ (p_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^k} : p_{\mathbf{i}} \geq 0, \sum_{\mathbf{i} \in \mathbb{N}^k} p_{\mathbf{i}} = 1 \right\}.$$

The set Δ_k together with the 1-norm $\|\cdot\|_1$ defined by

$$\|\mathbf{p} - \mathbf{q}\|_1 = \sum_{\mathbf{i} \in \mathbb{N}^k} |p_{\mathbf{i}} - q_{\mathbf{i}}|.$$

is a compact metric space.

On the one hand we clearly have that any vector $\Pi_k(\omega, n)$ of frequencies of blocks of digits of length k belongs to Δ_k . On the other hand any probability vector needs to be shift invariant (*cf.* Volkmann [22] or Olsen [17, 19]). Therefore we define the subsimplex of shift invariant probability vectors S_k by

$$S_k := \left\{ (p_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^k} : p_{\mathbf{i}} \geq 0, \sum_{\mathbf{i} \in \mathbb{N}^k} p_{\mathbf{i}} = 1, \sum_{i \in \mathbb{N}} p_{i\mathbf{i}} = \sum_{i \in \mathbb{N}} p_{i\mathbf{i}} \text{ for all } \mathbf{i} \in \mathbb{N}^{k-1} \right\}.$$

Now we want to focus on two different examples of non-normal sets. Our first one is the set of extremely non-normal numbers \mathbb{E} introduced by Olsen [17]. We call a number $x \in \mathbb{I}$ extremely non- k -normal if each probability vector $\mathbf{p} \in S_k$ is an accumulation point of the sequence of vectors of frequencies $(\Pi_k(x, n))_{n \in \mathbb{N}}$. Furthermore we call $x \in \mathbb{I}$ extremely non-normal if it is extremely non- k -normal for every $k \geq 1$. Then Olsen [17] could prove the following

Theorem 1.1 ([17, Theorem 1]). *The set of extremely non-normal numbers is residual.*

Our second example are continued fractions with maximal frequency oscillation. We denote by $A(x, \mathbf{b})$ the set of all accumulation points of the sequence $(\Pi(x, \mathbf{b}, n))_{n \in \mathbb{N}}$. Furthermore we set

$$A(\mathbf{b}) = \bigcup_{x \in \mathbb{I}} A(x, \mathbf{b}).$$

Then the set of continued fractions with maximal frequency oscillation \mathbb{F} is defined by

$$\mathbb{F} = \{x \in \mathbb{I} : A(\mathbf{b}) = A(x, \mathbf{b}) \text{ for all } \mathbf{b}\}.$$

Liao, Ma and Wang [13] could show the following

Theorem 1.2 ([13, Theorem 1.1]). *The set of continued fractions with maximal frequency oscillation is residual.*

These examples explain the dichotomy between normal and non-normal numbers. Since Lebesgue almost all numbers are normal, they are natural elements in the sense of probability. Whereas the set of non-normal numbers is residual and therefore they are natural elements in the sense of topology. This dichotomy has been exploited amongst others for the q -adic number systems by Alberverio, Hyde, Laschos, Olsen, Petrykiewicz, Pratsiovytyi, Shaw, Torbin and Volkmann in [1, 2, 9, 19, 22], for the continued fractions by Olsen [17] and for iterated function systems by Beak and Olsen [3].

In the present paper we want to investigate two main extensions of these examples. First we want to generalize the continued fraction expansions to infinite Markov partitions. A similar step has already been made for finite Markov partitions in another paper by the author [16]. Motivated by considerations of Hyde *et al.* [9] we want to investigate the Baire category of Cesàro variants of the sets of extremely non-normal numbers and numbers with maximal oscillation frequency.

2. DEFINITIONS AND STATEMENT OF RESULTS

We want to start by taking a step back and consider general dynamical systems. In our notation we mainly follow Chapter 6 of [14]. Let M be a metric space and $\phi : M \rightarrow M$ be a continuous map, then we call the pair (M, ϕ) a dynamical system. As above we need some “digits”. Let $\mathcal{P} := \{P_1, P_2, \dots\}$ be an infinite family of disjoint open sets. Then we call \mathcal{P} a topological partition (of M) if M is the union of the closures $\overline{P_i}$ for $i \geq 1$, *i.e.*

$$M = \bigcup_{i \in \mathbb{N}} \overline{P_i}.$$

For the rest of the paper let us assume that a dynamical system (M, ϕ) together with an infinite topological partition \mathcal{P} is given. Now we want to take a closer look at the underlying symbolic dynamical system. Without loss of generality we may denote by \mathbb{N} the alphabet corresponding to the partition \mathcal{P} . Furthermore let \mathbb{N}^k be the set of words of length k and

$$\mathbb{N}^* = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$$

be the set of finite words. Finally we denote by $\mathbb{N}^{\mathbb{N}}$ the set of infinite words over \mathbb{N} .

For an infinite word $\omega = \omega_1 \omega_2 \omega_3 \dots \in \mathbb{N}^{\mathbb{N}}$ and a positive integer n we denote by $\omega|n = \omega_1 \omega_2 \dots \omega_n$ its truncation to the n th place. Furthermore for a given finite word $\omega \in \mathbb{N}^*$ we denote by $[\omega] \subset \mathbb{N}^{\mathbb{N}}$ the cylinder set consisting of all infinite words starting with the same letters as ω , *i.e.*

$$[\omega] := \left\{ \gamma \in \mathbb{N}^{\mathbb{N}} : \gamma| |\omega| = \omega \right\}.$$

Now we want to describe the shift space that is generated. Therefore we call an infinite word $\omega = a_1 a_2 a_3 \dots \in \mathbb{N}^{\mathbb{N}}$ allowed for (\mathcal{P}, ϕ) if

$$\bigcap_{k=1}^{\infty} \phi^{-k}(P_{a_k}) \neq \emptyset.$$

Let $\mathcal{L}_{\mathcal{P}, \phi}$ be the set of allowed words. Then $\mathcal{L}_{\mathcal{P}, \phi}$ is a language and there is a unique shift space $X_{\mathcal{P}, \phi} \subseteq \mathbb{N}^{\mathbb{N}}$, whose language is $\mathcal{L}_{\mathcal{P}, \phi}$. We call $X_{\mathcal{P}, \phi} \subseteq \mathbb{N}^{\mathbb{N}}$ the one-sided symbolic dynamical

system corresponding to (\mathcal{P}, ϕ) . Finally for each $\omega = a_1 a_2 a_3 \dots \in X_{\mathcal{P}, \phi}$ and $n \geq 0$ we denote by $D_n(\omega)$ the cylinder set of order n corresponding to ω in M , *i.e.*,

$$D_n(\omega) := \bigcap_{k=0}^n \phi^{-k}(P_{a_k}) \subseteq M.$$

Now we can state the definition of an infinite Markov partition.

Definition 2.1. Let (M, ϕ) be a dynamical system and $\mathcal{P} = \{P_1, P_2, P_3, \dots\}$ be an infinite topological partition of M . Then we call \mathcal{P} an infinite Markov partition if the generated shift space $X_{\mathcal{P}, \phi}$ is of finite type and for every $\omega \in X_{\mathcal{P}, \phi}$ the intersection $\bigcap_{n=0}^{\infty} \overline{D_n(\omega)}$ consists of exactly one point.

After introducing all the necessary ingredients we want to link the introduced concept of infinite Markov partitions with the continued fraction expansion and Lüroth series (*cf.* Dajani and Kraaikamp [6]).

Example 2.2. The dynamical system $([0, 1], T)$, where T is the Gauss map

$$Tx = \begin{cases} \frac{1}{x} - \lfloor \frac{1}{x} \rfloor & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

together with the infinite topological partition

$$\mathcal{P} := \left\{ \left(\frac{1}{2}, 1 \right), \left(\frac{1}{3}, \frac{1}{2} \right), \left(\frac{1}{4}, \frac{1}{3} \right), \dots \right\}$$

provides the continued fraction expansion.

In particular, if we set $x_1 := x$ and $x_{k+1} := Tx_k$ for $k \geq 1$ then $a_k(x) = \lfloor \frac{1}{x_k} \rfloor$ for $k \geq 1$ is the continued fraction expansion of x from the introduction.

There are several extensions of the continued fraction expansion like continued fractions from below, nearest integers continued fractions, α -continued fractions, Rosen continued fractions and combination of those. For criteria such that a number is normal with respect to different continued fractions expansions we refer the reader to a paper by Kraaikamp and Nakada [10].

Example 2.3. Let $\phi : [0, 1) \rightarrow [0, 1)$ be defined by

$$\phi(x) = \begin{cases} n(n+1)x - n, & x \in \left[\frac{1}{n+1}, \frac{1}{n} \right), \\ 0 & x = 0. \end{cases}$$

Then the pair $([0, 1), \phi)$ together with the infinite topological partition

$$\mathcal{P} := \left\{ \left(\frac{1}{2}, 1 \right), \left(\frac{1}{3}, \frac{1}{2} \right), \left(\frac{1}{4}, \frac{1}{3} \right), \dots \right\}$$

provides the Lüroth series [15].

Under some mild restrictions one can replace the intervals by arbitrary ones in order to get the generalized Lüroth series (*cf.* Chapter 2.3 of [6]).

Before pulling over the definition of normal and non-normal numbers, we note that in contrast to the survey of Barat *et al.* [4] we did not use fibered systems for the definition of the dynamical system. The reason lies in the concrete treatment of the border in the case of Markov partitions. In partiucalar, when considering these partitions it is clear that the sets \mathcal{P}_i are all open sets, whereas this is a priory not clear in the definition of fibered

systems by Schweiger [21]. This plays a key role in the following analysis of the one-to-one correspondence between the infinite word and the corresponding element of M . By the definition of a Markov partition we have that every $\omega \in X_{\mathcal{P},\phi}$ maps to a unique element $x \in M$. However, the converse need not be true. Let us consider the continued fractions expansion (Exemple 2.2). Then the rational $\frac{1}{4}$ has two expansions, namely $[4]$ and $[3, 1]$. One observes that this ambiguity originates from the intersections $\overline{P_i} \cap \overline{P_j}$ for $i \neq j$ (which means from the borders of $\overline{P_i}$). Thus we concentrate on the inner points, which provide us with an infinite and unique expansion. Let

$$U = \bigcup_{i=1}^{\infty} P_i,$$

which is an open and dense ($\overline{U} = M$) set. Then for each $n \geq 1$ the set

$$U_n = \bigcap_{k=0}^{n-1} \phi^{-k}(U),$$

is open and dense in M . Thus by the Baire Category Theorem, the set

$$(2.1) \quad U_{\infty} = \bigcap_{n=1}^{\infty} U_n$$

is dense. Since $M \setminus U_{\infty}$ is the countable union of nowhere dense sets it suffices to show that a set is residual in U_{∞} in order to show that in fact it is residual in M .

Since the definition of normal and thus non-normal numbers involves the expansions of the elements in M we need the map $\pi_{\mathcal{P},\phi} : X_{\mathcal{P},\phi} \rightarrow M$ defined by

$$\{\pi_{\mathcal{P},\phi}(\omega)\} = \bigcap_{n=1}^{\infty} \overline{D_n(\omega)}.$$

Since $\pi_{\mathcal{P},\phi}$ is bijective on U_{∞} we may call ω **the** symbolic expansion of x if $\pi_{\mathcal{P},\phi}(\omega) = x$. Thus in the following we will silently suppose that $x \in U_{\infty}$.

After defining the environment we want to pull over the definitions of normal and non-normal numbers to the symbolic dynamical system. To this end let $\mathbf{b} \in \mathbb{N}^k$ be a block of letters of length k and $\omega = a_1 a_2 a_3 \dots \in X_{\mathcal{P},\phi}$ be the symbolic representation of an element. Then we write

$$P(\omega, \mathbf{b}, n) = |\{0 \leq i < n : a_{i+1} = b_1, \dots, a_{i+k} = b_k\}|$$

for the frequency of the block \mathbf{b} among the first n letters of ω . In the same manner as above let

$$P_k(\omega, n) = (P(\omega, \mathbf{b}), n)_{\mathbf{b} \in \mathbb{N}^k}$$

be the vector of all frequencies of blocks \mathbf{b} of length k among the first n letters of ω .

Let μ be a given ϕ -invariant probability measure on X and $\omega \in X$. Then we call the measure μ associated to ω if there exists a infinite sub-sequence F of \mathbb{N} such that for any block $\mathbf{b} \in \Sigma^k$

$$\lim_{\substack{n \rightarrow \infty \\ n \in F}} P(\omega, \mathbf{b}, n) = \mu([\mathbf{b}]).$$

Furthermore, we call ω a generic point for μ if we can take $F = \mathbb{N}$: then μ is the only measure associated with ω . If μ is the maximal measure, then we call ω normal.

An application of Birkhoff's ergodic theorem yields for μ being ergodic that almost all numbers $\omega \in X_{\mathcal{P},\phi}$ are normal. In both Examples 2.2 and 2.3 we have that the system is

intrinsically ergodic, which means that there exists a unique maximal ergodic measure μ (cf. Chapter 3.1.2 of [6]).

Now we turn our attention to non-normal numbers. As in the paper of Hyde *et al.* [9] we extend our considerations to Cesàro averages of the frequencies. The idea behind is that if the vector of frequencies has no accumulation point, his Cesàro average might have one. To this end for a fixed block $\mathbf{b} = b_1 \dots b_k \in \mathbb{N}^k$ let

$$P^{(0)}(\omega, \mathbf{b}, n) = P(\omega, \mathbf{b}, n).$$

For $r \geq 1$ we recursively define

$$P^{(r)}(\omega, \mathbf{b}, n) = \frac{\sum_{j=1}^n P^{(r-1)}(\omega, \mathbf{b}, j)}{n}$$

to be the r th iterated Cesàro average of the frequency of the block of digits \mathbf{b} under the first n digits. Furthermore we define by

$$P_k^{(r)}(\omega, n) := \left(P^{(r)}(\omega, \mathbf{b}, n) \right)_{\mathbf{b} \in \mathbb{N}^k}$$

the vector of r th iterated Cesàro averages. As above we are interested in the accumulation points. Thus similar to above let $A_k^{(r)}(\omega)$ denote the set of accumulation points of the sequence $(P_k^{(r)}(\omega, n))_n$ with respect to $\|\cdot\|_1$, *i.e.*

$$A_k^{(r)}(\omega) := \left\{ \mathbf{p} \in \Delta_k : \mathbf{p} \text{ is an accumulation point of } (P_k^{(r)}(\omega, n))_n \right\}.$$

We will denote the set of extremely non- k -normal numbers of M by $\mathbb{E}_k^{(0)}$. Similarly for $r \geq 1$ and $k \geq 1$ we denote by $\mathbb{E}_k^{(r)}$ the set of r th iterated Cesàro extremely non- k -normal numbers of M . Furthermore for $r \geq 1$ we denote by $\mathbb{E}^{(r)}$ the set of r th iterated Cesàro extremely non-normal numbers and by \mathbb{E} the set of completely Cesàro extremely non-normal numbers, *i.e.*

$$\mathbb{E}^{(r)} = \bigcap_k \mathbb{E}_k^{(r)} \quad \text{and} \quad \mathbb{E} = \bigcap_r \mathbb{E}^{(r)} = \bigcap_{r,k} \mathbb{E}_k^{(r)}.$$

Then our result is the following

Theorem 2.4. *Let $k \geq 1$ and $r \geq 0$ be integers. Furthermore let $\mathcal{P} = \{P_1, P_2, \dots\}$ be an infinite Markov partition for (M, ϕ) . Suppose that the generated shift space $X_{\mathcal{P}, \phi}$ is the one-sided full-shift. Then the set $\mathbb{E}_k^{(r)}$ is residual.*

Remark 2.5. We note that the same holds true for $X_{\mathcal{P}, \phi}$ being a one-sided shift of finite type. In fact the only change is a replacement of the definition of Z_n and of Lemma 3.1 (cf. Olsen [18] and Olsen and Winter [20]).

After considering extremely non-normal numbers we want to turn our attention towards numbers with maximal oscillation frequency. Similarly to above, for $r \geq 0$, we denote by $A^{(r)}(\omega, \mathbf{b})$ the set of all accumulation points of the sequence $(P^{(r)}(\omega, \mathbf{b}, n))_{n \in \mathbb{N}}$. Furthermore we set

$$A^{(r)}(\mathbf{b}) = \bigcup_{\omega \in U_\infty} A^{(r)}(\omega, \mathbf{b}).$$

Then the set of numbers with r -th iterated Cesàro maximal frequency oscillation $\mathbb{F}^{(r)}$ is defined by

$$\mathbb{F}^{(r)} = \left\{ \omega \in U_\infty : A^{(r)}(\mathbf{b}) = A^{(r)}(\omega, \mathbf{b}) \text{ for all } \mathbf{b} \in \mathbb{N}^* \right\}.$$

Our result is a generalization of Theorem 1 of Liao, Ma and Wang [13].

Theorem 2.6. *Let $r \geq 0$ be an integer and let $\mathcal{P} = \{P_1, P_2, \dots\}$ be an infinite Markov partition for (M, ϕ) . Suppose that the generated shift space $X_{\mathcal{P}, \phi}$ is the one-sided full-shift. Then $\mathbb{F}^{(r)}$ is residual.*

In the subsequent sections we will proof the two Theorems 2.4 and 2.6. We will start with a general section on properties of words which we need in the proof of Theorem 2.4 and which are interesting on their own. Then we will show Theorem 2.4. Finally in Section 5 we will prove Theorem 2.6 by showing that $\mathbb{E}^{(r)} \subset \mathbb{F}^{(r)}$.

3. PRELIMINARIES ON WORDS

First of all we want to reduce the infinite problem to a finite one. Thus instead of considering S_k as such we concentrate on those probability vectors that only put weight on a finite set of digits. Thus let

$$(3.1) \quad S_{k,N} = \left\{ (p_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^k} : \begin{array}{l} p_{\mathbf{i}} \geq 0, \sum_{\mathbf{i} \in \mathbb{N}^k} p_{\mathbf{i}} = 1, \sum_{\mathbf{i} \in \mathbb{N}} p_{\mathbf{i}\mathbf{i}} = \sum_{\mathbf{i} \in \mathbb{N}} p_{\mathbf{i}\mathbf{i}} \text{ for all } \mathbf{i} \in \mathbb{N}^{k-1} \\ p_{\mathbf{i}} = 0 \text{ for } \mathbf{i} \in \mathbb{N}^k \setminus \{1, \dots, N\}^k \end{array} \right\}$$

be the set of shift invariant probability vectors, where only the first N digits are weighted. Furthermore let

$$(3.2) \quad S_k^* = \bigcup_{N \geq 1} S_{k,N}$$

be the union of all probability vectors over a finite alphabet.

Since S_k^* is a dense and separable subset of S_k , we may concentrate on a dense sequence $(\mathbf{q}_{k,m})_m$ in S_k^* . We fix $\mathbf{q} = \mathbf{q}_{k,m}$ throughout the rest of this section. Then $\mathbf{q} \in S_{k,N}$ for some $N \geq 1$, such that $\mathbf{q}_{\mathbf{i}} = 0$ for $\mathbf{i} \in \mathbb{N}^k \setminus \{1, \dots, N\}^k$. For $n \geq 1$ we put

$$Z_n = Z_n(\mathbf{q}, N, k) = \left\{ \omega \in \bigcup_{\ell \geq knN^k} \{1, \dots, N\}^\ell \mid \|P_k(\omega) - \mathbf{q}\|_1 \leq \frac{1}{n} \right\}.$$

Since \mathbf{q} , N and k will be fixed we may omit them throughout the rest of this section.

The main idea consists now in the construction of a word having the desired frequencies. In particular, for a given word ω we want to show that we can add sufficiently many copies of any word from Z_n to get a word with the desired properties. To this end we first need, that there is at least one word in Z_n , i.e. that Z_n is not empty.

Lemma 3.1 ([17, Lemma 2.4]). *For all $n \geq 1$, $\mathbf{q} \in S_k^*$, $N \in \mathbb{N}$ and $k \in \mathbb{N}$ we have $Z_n(\mathbf{q}, N, k) \neq \emptyset$.*

Now we may construct our word by adding arbitrary many copies of an element of Z_n .

Lemma 3.2. *Let N, n, t be positive integers and $\mathbf{q} \in S_{k,N}$. Furthermore let $\omega = \omega_1 \dots \omega_t \in \mathbb{N}^t$ be a word of length t and let $M = \max_{1 \leq i \leq t} \omega_i$ be the maximal “digit” in ω . Then, for any $\gamma \in Z_n(\mathbf{q}, N, k)$ and any*

$$(3.3) \quad \ell \geq L := t + |\gamma| \max \left(n, \frac{t}{k} \max \left(1, \frac{M^k}{N^k} \right) \right)$$

we get that

$$\|P_k(\omega\gamma^*, \ell) - \mathbf{q}\|_1 \leq \frac{6}{n}.$$

Proof. We set $s := |\gamma|$ and

$$\sigma = \omega\gamma^*|\ell.$$

Furthermore we set q and $0 \leq r < s$ such that $m = t + qs + r$. Since an occurrence can happen in ω , in γ , somewhere in between or at the end, for every $\mathbf{i} \in \mathbb{N}^k$ we clearly have that

$$\frac{qs}{\ell} P(\gamma, \mathbf{i}) \leq P(\sigma, \mathbf{i}) \leq \frac{qs P(\gamma, \mathbf{i})}{\ell} + \frac{t + q(k-1) + r}{\ell}.$$

Now we concentrate on the occurrences in multiples of γ and show that we may neglect those outside of γ , i.e.,

$$\|P_k(\sigma) - \mathbf{q}\|_1 \leq \left\| P_k(\sigma) - \frac{qs}{\ell} P_k(\gamma) \right\|_1 + \left\| \frac{qs}{\ell} P_k(\gamma) - \mathbf{q} \right\|_1.$$

We will estimate both parts separately. For the first one we get that

$$\begin{aligned} \left\| P_k(\sigma) - \frac{qs}{\ell} P_k(\gamma) \right\|_1 &= \sum_{\mathbf{i} \in \{1, \dots, N\}^k} \left| P(\sigma, \mathbf{i}) - \frac{qs}{\ell} P(\gamma, \mathbf{i}) \right| + \sum_{\mathbf{i} \in \mathbb{N}^k \setminus \{1, \dots, N\}^k} \left| P(\sigma, \mathbf{i}) - \frac{qs}{\ell} P(\gamma, \mathbf{i}) \right| \\ &\leq \sum_{\mathbf{i} \in \{1, \dots, N\}^k} \frac{t + qk + s}{\ell} + \sum_{\substack{\mathbf{i} \in \mathbb{N}^k \setminus \{1, \dots, N\}^k \\ P(\omega, \mathbf{i}) \neq 0}} \frac{t}{\ell} \\ &\leq N^k \frac{t + qk}{qnkN^k} + \frac{1}{q} + M^k \frac{t}{qnkN^k} \\ &= \frac{1}{n} + (c + 1) \frac{1}{q}, \end{aligned}$$

where we have used that $\ell \geq qs \geq qnkN^k$ and written

$$c = \frac{t}{nk} \left(1 + \frac{M^k}{N^k} \right).$$

For the second part we get that

$$\begin{aligned} \left\| \frac{qs}{\ell} P_k(\gamma) - \mathbf{q} \right\|_1 &\leq \left\| \frac{qs}{\ell} P_k(\gamma) - P_k(\gamma) \right\|_1 + \|P_k(\gamma) - \mathbf{q}\|_1 \\ &\leq qs \left| \frac{1}{\ell} - \frac{1}{qs} \right| + \frac{1}{n} \\ &\leq \frac{t}{\ell} + \frac{1}{n} \leq \frac{t}{qnkN^k} + \frac{1}{n}. \end{aligned}$$

Putting these together yields

$$\|P_k(\sigma) - \mathbf{q}\|_1 \leq \frac{1}{n} + (c + 1) \frac{1}{q} + \frac{t}{qnkN^k} + \frac{1}{n}.$$

By our assumptions on the size of ℓ in (3.3) this proves the lemma. \square

4. PROOF OF THEOREM 2.4

The standard method of proof is to construct a subset E of $\mathbb{E}_k^{(r)}$ which is easier to handle and already residual. In our construction of the set E we mainly follow the ideas of Hyde *et al.* [9]. We start by recursively defining the functions φ_m for $m \geq 1$ by $\varphi_1(x) = 2^x$ and $\varphi_m(x) = \varphi_1(\varphi_{m-1}(x))$ for $m \geq 2$. Furthermore we set $\mathbb{D} = (\mathbb{Q}^{(\mathbb{N}^k)} \cap S_k^*)$. Since \mathbb{D} is countable and dense in S_k^* and therefore dense in S_k we may concentrate on the probability vectors $\mathbf{q} \in \mathbb{D}$.

Now we say that a sequence $(\mathbf{x}_n)_n$ in $\mathbb{R}^{(\mathbb{N}^k)}$ has property P if for all $\mathbf{q} \in \mathbb{D}$, $m \in \mathbb{N}$, $i \in \mathbb{N}$, and $\varepsilon > 0$, there exists a $j \in \mathbb{N}$ satisfying:

- (1) $j \geq i$,
- (2) $j/2^j < \varepsilon$,
- (3) if $j < n < \varphi_m(j)$ then $\|\mathbf{x}_n - \mathbf{q}\|_1 < \varepsilon$.

Then we define our set E to consist of all frequency vectors having property P , *i.e.*

$$E = \{x \in U_\infty : (P_k^{(0)}(x; n))_{n=1}^\infty \text{ has property } P\}.$$

We will proceed in three steps showing that

- (1) E is residual,
- (2) if $(P^{(r)}(x; n))_{n=1}^\infty$ has property P , then also $(P^{(r+1)}(x; n))_{n=1}^\infty$ has property P , and
- (3) $E \subseteq \mathbb{E}_k^{(r)}$.

Lemma 4.1. *The set E is residual.*

Proof. For fixed $h, m, i \in \mathbb{N}$ and $\mathbf{q} \in \mathbb{D}$, we say that a sequence $(\mathbf{x}_n)_n$ in $\mathbb{R}^{\mathbb{N}^k}$ has property $P_{h,m,\mathbf{q},i}$ if for every $\varepsilon > 1/h$, there exists $j \in \mathbb{N}$ satisfying:

- (1) $j \geq i$,
- (2) $j/2^j < \varepsilon$,
- (3) if $j < n < \varphi_m(2^j)$, then $\|\mathbf{x}_n - \mathbf{q}\|_1 < \varepsilon$.

Now let $E_{h,m,\mathbf{q},i}$ be the set of all points whose frequency vector satisfies property $P_{h,m,\mathbf{q},i}$, *i.e.*

$$E_{h,m,\mathbf{q},i} := \left\{ x \in U_\infty : (P_k^{(0)}(x; n))_{n=1}^\infty \text{ has property } P_{h,m,\mathbf{q},i} \right\}.$$

Obviously we have that

$$E = \bigcap_{h \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcap_{\mathbf{q} \in \mathbb{D}} \bigcap_{i \in \mathbb{N}} E_{h,m,\mathbf{q},i}.$$

Thus it remains to show, that $E_{h,m,\mathbf{q},i}$ is open and dense.

- (1) $E_{h,m,\mathbf{q},i}$ **is open.** Let $x \in E_{h,m,\mathbf{q},i}$, then there exists a $j \in \mathbb{N}$ such that $j \geq i$, $j/2^j < 1/h$, and if $j < n < \varphi_m(2^j)$, then

$$\left\| P_k^{(1)}(x; n) - \mathbf{q} \right\|_1 < 1/h.$$

Let $\omega \in X$ be such that $x = \pi(\omega)$ and set $t := \varphi_m(2^j)$. Since $D_t(\omega)$ is open, there exists a $\delta > 0$ such that the ball $B(x, \delta) \subseteq D_t(\omega)$. Furthermore, since by definition all $y \in D_t(\omega)$ have their first t digits the same as x , we get that

$$B(x, \delta) \subseteq D_t(\omega) \subseteq E_{h,m,\mathbf{q},i}.$$

- (2) $E_{h,m,\mathbf{q},i}$ **is dense**. Let $x \in U_\infty$ and $\delta > 0$. We must find $y \in B(x, \delta) \cap E_{h,m,\mathbf{q},i}$.

Let $\omega \in X$ be such that $x = \pi(\omega)$. Since $\text{diam } \overline{D_t(\omega)} \rightarrow 0$ for $t \rightarrow \infty$ and $x \in D_t(\omega)$ for $t \geq 1$ there exists a t such that $D_t(\omega) \subset B(x, \delta)$. Let $\sigma = \omega|t$ be the first t digits of x .

Now, an application of Lemma 3.1 with $n = 6h$ yields that there exists a finite word γ such that

$$\|P_k(\gamma) - \mathbf{q}\|_1 \leq \frac{1}{6h}.$$

Let $\varepsilon \geq \frac{1}{h}$ and L be as in the statement of Lemma 3.2. Then we choose j such that

$$\frac{j}{2^j} < \varepsilon \quad \text{and} \quad j \geq \max(L, i).$$

An application of Lemma 3.2 with $n = 6h$ then gives us that

$$\|P_k(\sigma\gamma^*|j) - \mathbf{q}\|_1 \leq \frac{6}{n} = \frac{1}{h} \leq \varepsilon.$$

Thus we choose $y \in D_j(\sigma\gamma^*)$. Then on the one hand $y \in D_j(\sigma\gamma^*) \subset D_t(\omega) \subset B(x, \delta)$ and on the other hand $y \in D_j(\sigma\gamma^*) \subset E_{h,m,\mathbf{q},i}$.

It follows that E is the countable intersection of open and dense sets and therefore E is residual in U_∞ . \square

Lemma 4.2. *Let $\omega \in X_{\mathcal{P},\phi}$. If $(P^{(r)}(\omega, n))_{n=1}^\infty$ has property P , then also $(P^{(r+1)}(\omega, n))_{n=1}^\infty$ has property P .*

This is Lemma 2.2 of [9]. However, the proof is short so we present it here for completeness.

Proof. Let $\omega \in X_{\mathcal{P},\phi}$ be such that $(P_k^{(r)}(\omega, n))_{n=1}^\infty$ has property P , and fix $\varepsilon > 0$, $\mathbf{q} \in \mathbb{D}$, $i \in \mathbb{N}$ and $m \in \mathbb{N}$. Since $(P_k^{(r)}(\omega, n))_{n=1}^\infty$ has property P , there exists $j' \in \mathbb{N}$ with $j' \geq i$, $j'/2^{j'} < \varepsilon/3$, and such that for $j' < n < \varphi_{m+1}(2^{j'})$ we have that $\|P_k^{(r)}(\omega, n) - \mathbf{q}\|_1 < \varepsilon/3$.

We set $j = 2^{j'}$ and show that $(P_k^{(r+1)}(\omega, n))_{n=1}^\infty$ has property P with this j . For all $j < n < \varphi_m(2^j)$ (i.e. $2^{j'} < n < \varphi_{m+1}(2^{j'})$), we have

$$\begin{aligned} \|P_k^{(r+1)}(\omega, n) - \mathbf{q}\|_1 &= \left\| \frac{P_k^{(r)}(\omega, 1) + P_k^{(r)}(\omega, 2) + \cdots + P_k^{(r)}(\omega, n)}{n} - \mathbf{q} \right\|_1 \\ &= \left\| \frac{P_k^{(r)}(\omega, 1) + P_k^{(r)}(\omega, 2) + \cdots + P_k^{(r)}(\omega, j')}{n} \right. \\ &\quad \left. + \frac{P_k^{(r)}(\omega, j'+1) + P_k^{(r)}(\omega, j'+2) + \cdots + P_k^{(r)}(\omega, n) - (n-j')\mathbf{q}}{n} - \frac{j'\mathbf{q}}{n} \right\|_1 \\ &\leq \frac{\|P_k^{(r)}(\omega, 1) + P_k^{(r)}(\omega, 2) + \cdots + P_k^{(r)}(\omega, j')\|_1}{n} \\ &\quad + \frac{\|P_k^{(r)}(\omega, j'+1) - \mathbf{q}\|_1 + \cdots + \|P_k^{(r)}(\omega, n) - \mathbf{q}\|_1}{n} - \frac{\|j'\mathbf{q}\|_1}{n} \\ &\leq \frac{j'}{n} + \frac{\varepsilon(n-j')}{3n} + \frac{j'}{n} \leq \frac{j'}{2^{j'}} + \frac{\varepsilon}{3} + \frac{j'}{2^{j'}} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Lemma 4.3. *The set E is a subset of $\mathbb{E}_k^{(r)}$.*

Proof. We will show, that for any $x \in E$ we also have $x \in \mathbb{E}_k^{(r)}$. To this end, let $x \in E$ and $\omega \in X_{\mathcal{P},\phi}$ be the symbolic expansion of x , i.e. $x = \pi(\omega)$. Since $(P_k^{(0)}(\omega, n))_n$ has property P , by iterating Lemma 4.2 we get that $(P_k^{(r)}(\omega, n))_n$ has property P .

Thus it suffices to show that \mathbf{p} is an accumulation point of $(P_k^{(r)}(\omega, n))_n$ for any $\mathbf{p} \in S_k$. Therefore we fix $h \in \mathbb{N}$ and, since \mathbb{D} is dense in S_k , we find a $\mathbf{q} \in \mathbb{D}$ such that

$$\|\mathbf{p} - \mathbf{q}\|_1 < \frac{1}{h}.$$

Since $(P_k^{(r)}(\omega, n))_n$ has property P for any $m \in \mathbb{N}$ we find $j \in \mathbb{N}$ with $j \geq h$ and such that if $j < n < \varphi_m(2^j)$ then $\|P_k^{(r)}(\omega, n) - \mathbf{q}\|_1 < \frac{1}{h}$. Hence let n_h be any integer with $j < n_h < \varphi_m(2^j)$, then

$$\|P_k^{(r)}(\omega, n_h) - \mathbf{q}\|_1 < \frac{1}{h}.$$

Thus each n_h in the sequence $(n_h)_h$ satisfies

$$\|\mathbf{p} - P_k^{(r)}(\omega, n_h)\|_1 \leq \|\mathbf{p} - \mathbf{q}\|_1 + \|P_k^{(r)}(\omega, n_h) - \mathbf{q}\|_1 < \frac{2}{h}.$$

Since $n_h > h$ we may extract an increasing sub-sequence $(n_{h_u})_u$ such that $P_k^{(r)}(\omega, n_{h_u}) \rightarrow \mathbf{p}$ for $u \rightarrow \infty$. Thus \mathbf{p} is an accumulation point of $P_k^{(r)}(\omega, n)$, which proves the lemma. □

Proof of Theorem 2.4. Since by Lemma 4.1 E is residual in U_∞ and by Lemma 4.3 E is a subset of $\mathbb{E}_k^{(r)}$ we get that $\mathbb{E}_k^{(r)}$ is residual in U_∞ . Again we note that $M \setminus U_\infty$ is the countable union of nowhere dense sets and therefore $\mathbb{E}_k^{(r)}$ is also residual in M . □

5. PROOF OF THEOREM 2.6

Following the proof of Liao, Ma and Wang [13] it suffices to show that

$$\mathbb{E}^{(r)} \subset \mathbb{F}^{(r)}.$$

First the following lemma provides us with a suitable definition of $A^{(r)}(\omega, \mathbf{b})$.

Lemma 5.1. *Let $r \geq 0$ be an integer, $\omega \in X_{\mathcal{P},\phi}$ and $\mathbf{b} \in \mathbb{N}^*$. Then*

$$A^{(r)}(\omega, \mathbf{b}) = \left[\liminf_{n \rightarrow \infty} P^{(r)}(\omega, \mathbf{b}, n), \limsup_{n \rightarrow \infty} P^{(r)}(\omega, \mathbf{b}, n) \right].$$

Proof. It suffices to show that the gaps between two consecutive frequencies tend to zero, i.e.

$$\lim_{n \rightarrow \infty} (P^{(r)}(\omega, \mathbf{b}, n+1) - P^{(r)}(\omega, \mathbf{b}, n)) = 0.$$

For $r = 0$ a direct upper and lower estimate for the number of occurrences yields

$$\left| P^{(0)}(\omega, \mathbf{b}, n+1) - P^{(0)}(\omega, \mathbf{b}, n) \right| \leq \frac{1}{n+1}.$$

Since, for $i, j \geq 1$ and $r \geq 0$, we have that

$$\left| P^{(r)}(\omega, \mathbf{b}, i) - P^{(r)}(\omega, \mathbf{b}, j) \right| \leq 1,$$

we get by the definition of $P^{(r+1)}(\omega, \mathbf{b}, n)$ that

$$\left| P^{(r+1)}(\omega, \mathbf{b}, n+1) - P^{(r+1)}(\omega, \mathbf{b}, n) \right| \leq \frac{\sum_{j=1}^n |P^{(r)}(\omega, \mathbf{b}, n+1) - P^{(r)}(\omega, \mathbf{b}, j)|}{(n+1)n} \leq \frac{1}{n+1}.$$

□

Let $\mathbf{b} = b_1 b_2 \dots b_k \in \mathbb{N}^*$ be a word of length k . Then we denote by $\text{per}(\mathbf{b})$ the basic period of \mathbf{b} , i.e.

$$\text{per}(\mathbf{b}) := \min\{p \leq k : b_{p+j} = b_j \text{ for } 1 \leq j \leq k-p\}.$$

Furthermore we call the factor $\tilde{\mathbf{b}} := b_1 \dots b_{\text{per}(\mathbf{b})}$ the basic factor. Then we have the following

Lemma 5.2. *Let $r \geq 0$ be an integer and $\mathbf{b} \in \mathbb{N}^*$ be a finite word with basic period p and basic factor $\tilde{\mathbf{b}} = b_1 \dots b_p$. Then, for each $n \geq 2$,*

$$\lim_{n \rightarrow \infty} P^{(r)}(\tilde{\mathbf{b}}^*, \mathbf{b}, n) = \frac{1}{p}.$$

Proof. For $r = 0$ this is Lemma 2.2 of [13]. The case $r \geq 1$ follows, since

$$\lim_{n \rightarrow \infty} P^{(r)}(\tilde{\mathbf{b}}^*, \mathbf{b}, n) = \lim_{n \rightarrow \infty} P^{(r-1)}(\tilde{\mathbf{b}}^*, \mathbf{b}, n) = \dots = \lim_{n \rightarrow \infty} P^{(0)}(\tilde{\mathbf{b}}^*, \mathbf{b}, n) = \frac{1}{p}.$$

□

Now we have enough tools to state the proof of Theorem 2.6.

Proof of Theorem 2.6. Let $\omega \in \mathbb{E}^{(r)}$ and $\mathbf{b} = b_1 \dots b_k \in \mathbb{N}^*$ be a finite word with basic period p . Then Lemma 5.1 and Lemma 5.2 imply that

$$A^{(r)}(\mathbf{b}) = \left[0, \frac{1}{p}\right].$$

Therefore in order to prove that $\omega \in \mathbb{F}^{(r)}$ it suffices to show that 0 and $\frac{1}{p}$ are limit points of $(P^{(r)}(\omega, \mathbf{b}, n))_{n \in \mathbb{N}}$. Furthermore, since $\omega \in \mathbb{E}$, for any $\varepsilon > 0$ and $\mathbf{q} \in S_k$ we have $\|P_k^{(r)}(\omega, n) - \mathbf{q}\|_1$ for infinitely many n . Thus it suffices to find two suitable probability vectors \mathbf{q} providing the limit points 0 and $\frac{1}{p}$ for $(P^{(r)}(\omega, \mathbf{b}, n))_{n \in \mathbb{N}}$.

- **0 is limit point.** We chose a digit d which is bigger than any digit in \mathbf{b} , i.e. $d > \max\{w_i : 1 \leq i \leq k\}$. Then we define the probability vector $\mathbf{q} = (q_i)_{i \in \mathbb{N}^k}$ by

$$q_i = \begin{cases} 1 & \text{if } \mathbf{i} = \underbrace{d \dots d}_{k \text{ times}}, \\ 0 & \text{else.} \end{cases}$$

We clearly have that $\mathbf{q} \in S_k$. Since $P^{(r)}(\omega, \mathbf{b}, n) < \varepsilon$ infinitely often, we have that 0 is a limit point.

- **$\frac{1}{p}$ is limit point.** We note that $\gamma = \mathbf{b}^\infty$ is a periodic point with minimal period p under the map ϕ . Let μ be the periodic orbit measure, which has mass $\frac{1}{p}$ at each of the points $\{\gamma, \phi\gamma, \dots, \phi^{p-1}\gamma\}$. Then μ is shift-invariant and induces a shift-invariant probability vector

$$\mathbf{q} = (q_{\mathbf{b}})_{\mathbf{b} \in \mathbb{N}^k} = (\mu(\mathbf{b}))_{\mathbf{b} \in \mathbb{N}^k}.$$

Since $\mathbf{q} \in S_k$ and $q_{\mathbf{b}} = \mu(\mathbf{b}) = \frac{1}{p}$, we have $|P^{(r)}(\omega, \mathbf{b}, n) - \frac{1}{p}| < \varepsilon$ infinitely often.

Therefore $\frac{1}{p}$ is also a limit point.

□

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